PRIMORDIAL BLACK HOLES FROM INFLATIONARY MODELS WITH AND WITHOUT BROKEN SCALE INVARIANCE

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Abstract

We review the formalism of primordial black holes (PBHs) production and show that the mass variance at horizon crossing has been systematically overestimated in previous studies. We derive the correct expression. The difference is maximal at the earliest formation times and still very significant for PBH masses $\sim 10^{15}$g, an accurate estimate requiring numerical calculations. In particular, this would lead to weaker constraints on the spectral index n. We then derive constraints on inflationary models from the fact that primordial black holes must not overclose the Universe. This is done both for the scale-free case of the power spectrum studied earlier and for the case where a step in the mass variance is superimposed. In the former case we find various constraints on n, depending on the parameters. In the latter case these limits can be much more strengthened, so that one could find from an observational limit on n a constraint on the allowed height of the step.

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1 Introduction

Cosmology has now entered an exciting stage where observations of ever increasing accuracy can probe the remote past of our Universe. The present paradigm makes use of an inflationary stage of expansion in the very early Universe (see e.g. [1, 2, 3]). This solves some of the big problems of the old big bang cosmology in an elegant way, in particular the problems of horizon and flatness. But most importantly, and there lies its predictive power, it gives a possible solution to the crucial problem of where the primordial fluctuations leading to the observed large-scale structure (LSS) come from. In fact, they have their origin in the ubiquitous vacuum fluctuations. The seed of the LSS has been observed in the form of tiny fluctuations imprinted on the cosmological microwave background (CMB) at the time of decoupling. Each inflationary model makes precise predictions about the spectrum of its primordial fluctuations and this is how these models can be constrained by observations.

It was realized already some time ago that primordial fluctuations can lead to the formation of primordial black holes (PBHs) (see e.g. [4] for a review). For this, one needs a spectrum of primordial fluctuations whose statistics and amplitude are known. Inflationary models produce a spectrum of fluctuations also on scales relevant for the formation of PBHs. The statistics for PBH formation is usually taken as Gaussian for simplicity, which fits most of the inflationary models. One therefore possesses an additional tool to constrain inflationary models. The a priori interesting thing about PBHs is that they probe scales which are many orders of magnitude smaller than scales probed by large-scale structure (LSS) surveys and CMB angular anisotropy observations. In this sense it is analogous, even if less spectacular, to the primordial gravitational wave background of inflationary origin.

It was already shown that PBH formation can put constraints on the spectral index of the primordial density perturbations, $n \leq (1.23 - 1.25)$, see e.g. [5, 6, 7, 8]. Basically, these constraints come either from the evaporation of black holes (leading to a somewhat stronger limit) or from the fact that PBHs must not overclose the present Universe, i.e., from the requirement $\Omega_{PBH,0} < 1$. On mass scales $M \lesssim 10^{35}$ g, these constraints on n are even stronger than those coming from COBE. There are also constraints from the need not to overproduce PBHs during the preheating phase following inflation [9, 10]. Such constraints assume of course that the primordial spectrum is scale-free, an assumption that we shall relax here. Hence there are two attractive aspects in the study of PBHs using inflationary perturbations: it expands the range of scales that can be probed, and it can test possible features in the spectrum on these scales not probed by CMB and LSS observations.

A characteristic scale can be generated, for example, in two-fields inflation such as double inflation (see e.g. [11, 12]) or in single-field inflation with broken scale invariance (BSI) (see e.g. [13, 14]). Such a model was used recently in order to drastically cut the power on small scales in an attempt to explain the dearth of dwarf galaxies [15]. We want here to extend this study to the formation of PBHs. We are, therefore, interested in features on much smaller scales than those which we have considered previously, an exciting possibility which we seriously consider as there is no reason why the spectrum should be scale-free on all those scales relevant for PBH formation (though the actual occurrence of the scale-free case cannot be excluded).

Our paper is organized as follows. In Sect. II we start by reviewing the basic formalism for PBH formation. We then point out in detail how expressions for the mass variance used earlier have to be corrected in order to obtain accurate numbers. We then consider several concrete models, beginning with the usual scale-free case, then extending our study to the presence of a step in the mass variance. In Sect. III we then obtain constraints on the spectral index from the requirement that PBHs must not overclose the Universe. For the scale-free case we get various constraints on n, depending on the parameters of interest, while for the case of a scale-free spectrum with a superimposed step the constraint on n depends on the height of the step. Therefore, one can obtain constraints on this height from the existing limits on n. Finally, Sect. IV contains our conclusions.

2 PBH formation

2.1 Basic formalism

A spectrum of primordial density fluctuations can lead to the production of primordial black holes. Let us assume for simplicity that a PBH is formed when the density contrast averaged over a volume of the (linear) size of the Hubble radius satisfies $\delta_{min} \le \delta \le \delta_{max}$, and the PBH mass, M_{PBH}, is of the order of the "horizon mass" M_H, the mass contained inside the Hubble volume. Usually one takes $\delta_{min} = \frac{1}{3}$, $\delta_{max} = 1$ coming from semianalytic considerations [4]. Recent numerical calculations indicate, however, that PBHs can be formed over a much wider range of masses at a given formation time and that $\delta_{min} \approx 0.7$ [16, 17]. This is why we leave the limits open at this stage. For the purpose of studying the changes introduced by a characteristic scale in the primordial spectrum, it will be sufficient in the following to make the simplifying assumption that PBHs are formed with the mass M_H. We shall, however, also comment below on the changes introduced by the assumption that the PBH mass is only a certain fraction of M_H.

Each physical scale $R(t)$ is defined by some wavenumber k and evolves with time according to $R(t) = a(t)/k$. For a given physical scale $R(t)$, the "horizon" crossing time t_k – here we do not mean "horizon" crossing during inflation, but after inflation – is the time when this scale reenters the Hubble radius, which will inevitably happen after inflation for scales that are greater than the Hubble radius at the end of inflation. It is at this time t_k where the above condition for the density contrast holds. It will lead to the formation of a PBH with mass M_{PBH}, being approximately equal to $M_H(t_k)$. Clearly, there is a one-to-one correspondence between $R(t_k), M_H(t_k)$, and k. Of course we can also take this correspondence at any other initial time t_i and relate the physical quantities at both times t_i and t_k.

Generally, if the primordial fluctuations obey Gaussian statistics, the probability density $p_R(\delta)$, where δ is the density contrast averaged over a sphere of radius R, is given by

$$p_R(\delta) = \frac{1}{\sqrt{2\pi}\ \sigma(R)}\ e^{-\frac{\delta^2}{2\sigma^2(R)}}\ . \tag{1}$$

Here, the dispersion (mass variance) $\sigma^2(R) \equiv \left\langle \left(\frac{\delta M}{M}\right)^2_R \right\rangle$ is computed using a top-hat window function [2, 3],

$$\sigma^2(R) = \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}k\ k^2\ W^2_{TH}(kR)\ P(k)\ , \tag{2}$$

where $P(k)$ is the power spectrum. It is defined by $P(k) \equiv \langle |\delta_k|^2 \rangle$ in the case of finite volume. For inflation, one has to take instead the infinite-volume expression

$$\langle \delta_{\mathbf{k}} \delta^*_{\mathbf{k}'} \rangle \equiv P(k)\ \delta(\mathbf{k} - \mathbf{k}')\ , \tag{3}$$

where we assume isotropy of the ensemble. From a fundamental point of view, the averages $\langle \ldots \rangle$ refer to quantum expectation values; however, an effective quantum-to-classical transition is achieved during inflation [18, 19]. In the case of PBHs produced by inflationary perturbations, this quantum-to-classical transition is guaranteed for the masses of interest to us (see [20] and the remarks at the end of section 2.2).

The expression $W_{TH}(kR)$ stands for the Fourier transform of the top-hat window function divided by the probed volume $V_W = \frac{4}{3}\pi R^3$,

$$W_{TH}(kR) = \frac{3}{(kR)^3}(\sin kR - kR \cos kR)\ . \tag{4}$$

Therefore the probability $\beta(M_H)$ that a region of comoving size $R = k^{-1}$ has an averaged density contrast at horizon crossing t_k in the range $\delta_{min} \leq \delta \leq \delta_{max}$, is given by

$$\beta(M_H) = \frac{1}{\sqrt{2\pi}\,\sigma_H(t_k)} \int_{\delta_{min}}^{\delta_{max}} e^{-\frac{\delta^2}{2\sigma^2_H(t_k)}}\,\mathrm{d}\delta \approx \frac{\sigma_H(t_k)}{\sqrt{2\pi}\,\delta_{min}} e^{-\frac{\delta^2_{min}}{2\sigma^2_H(t_k)}}\ , \tag{5}$$

where $\sigma^2_H(t_k) := \sigma^2(R)|_{t_k}$, and the last approximation is valid for $\delta_{min} \gg \sigma_H(t_k)$, and $(\delta_{max} - \delta_{min}) \gg \sigma_H(t_k)$.

Important conclusions can be drawn from (5). Let us consider first the value of $\beta(M_H)$ today. We then have $\sigma^2_H(t_0) \simeq 10^{-10}$. So clearly the probability of forming a black hole today is extraordinarily small. This probability can be larger in the primordial universe if the power is increased when we go backwards in time, but usually the number will remain very small, $\beta(M_H) \ll 1$, at all times. This is due to the amplitude of $\delta^2_{min}/\sigma^2_H(t_k)$, on which $\beta(M_H)$ depends sensitively, see (5).

The expression (5) for $\beta(M_H)$ is usually interpreted as giving the probability that a PBH will be formed with a mass $M_{PBH} \geq M_H(t_k)$, i.e. greater or equal to the mass contained inside the Hubble volume when this scale reenters the Hubble radius. Strictly speaking, though, this is *not* true, since (5) does not take into account those regions that are underdense on a scale M_H, but nevertheless overdense on some larger scale. In the Press-Schechter formalism (see e.g. [2, 3]) this seems to be taken care of in some models by multiplying (5) with a factor 2. Fortunately, as emphasized in [7], in most cases $\beta(M_H)$ is a very rapidly falling function of mass, so this effect can be neglected. In this case, $\beta(M_H)$ *does* give the probability for PBH formation and thus also (at time t_k) the mass fraction of regions that will evolve into PBHs of mass greater or equal to M_H. This will also be the case for our models.

2.2 Improved calculation of the mass variance

As is clear from the previous subsection, one is interested in computing $\sigma^2(R)$ at horizon crossing using the power spectrum of the primordial fluctuations $P(k,t)$ of interest. Let us consider the simplest case which is usually considered when the power spectrum is scale-free and of the form $P(k,t) = A(t)\ k^n$. Then, the following equation is usually used to relate $\sigma_H(t_k)$ to the present value $\sigma_H(t_0)$, see e.g. [7],

$$\sigma^2_H(t_k) = \sigma^2_H(t_0)\ \left[\frac{M_H(t_0)}{M_H(t_{eq})}\right]^{\frac{n-1}{3}} \left[\frac{M_H(t_{eq}}{M_H(t_k)}\right]^{\frac{n-1}{2}}\ . \tag{6}$$

The expression (6) rests on the assumption that $\sigma_H(t_k)$ is well approximated by the integral (2), without window function but with a cut-off at the horizon scale. Consequently, the presence of the window function would be equivalent to the introduction of an effective cut-off at the horizon scale, leading to (recalling that $A(t_k) \propto k^{-4}$)

$$\sigma^2_H(t_k) \propto k^{n-1}\ . \tag{7}$$

However, for a scale-free primordial spectrum with $n \geq 1$, one recognizes that (2) is ultraviolet divergent! Hence it is clear that the integral is dominated by the contribution from small scales. Actually, even for $0.8 < n < 1$, the small scales will still contribute significantly to the integral and even dominate it for $n \approx 1$.

On the other hand, the quantities $k^3\ \Phi^2(k,t)$, or $\delta^2_H(k,t)$, defined as

$$\delta^2_H(k,t) \equiv \frac{(aH)^4}{k^4}\frac{k^3}{2\pi^2}\ P(k,t) \equiv \frac{2}{9\pi^2}\ k^3\ \Phi^2(k,t) \tag{8}$$

are time-independent on "superhorizon" scales (scales bigger than the Hubble radius, $k < aH$) in the cases that we consider here, being equal in very good approximation to their value at the horizon crossing time t_k. On these scales, these quantities behave as in (7). In addition, the constant of proportionality is different in the matter and radiation dominated stages. With t_{k_r} and t_{k_m} denoting times in the radiation and matter dominated stage, respectively, we get (cf. e.g. [11, 2])

$$\begin{aligned} k_r^3\ \Phi^2(k_r, t_{k_r}) &= \left(\frac{10}{9}\right)^2\ k_m^3\ \Phi^2(k_m, t_{k_m}) \left(\frac{k_r}{k_m}\right)^{n-1} , \\ \delta_H^2(k_r, t_{k_r}) &= \left(\frac{10}{9}\right)^2\ \delta_H^2(k_m, t_{k_m}) \left(\frac{k_r}{k_m}\right)^{n-1} . \end{aligned} \tag{9}$$

However, inspection of (5) shows that it is the quantity $\sigma_H(t_k)$ which is needed.

Actually (and fortunately), there is a natural upper cut-off in k-space for the power spectrum, namely k_e, corresponding to the Hubble radius at the end of inflation t_e. The lower limit can be taken zero if we assume that the number of e-folds during inflation amply solves the cosmological particle horizon problem. We thus have to consider only wavelengths bigger than k_e^{-1}. The relation between $\sigma_H(t_k)$ and $k^{\frac{3}{2}}\ \Phi(k, t_k)$, or $\delta_H(t_k)$, *depends on the time* t_k *and is model-dependent.* We have in general (with $k = (aH)|_{t_k}$)

$$\sigma_H^2(t_k) \equiv \alpha^2(k)\ \delta_H^2(k, t_k)\ , \tag{10}$$

where

$$\alpha^2(k) = \int_0^{\frac{k_e}{k}} x^{n+2}\ T^2(kx, t_k)\ W_{TH}^2(x)\ \mathrm{d}x\ . \tag{11}$$

The transfer function $T(k, t)$ is defined through

$$P(k, t) = \frac{P(0, t)}{P(0, t_i)}\ P(k, t_i)\ T^2(k, t)\ , \quad T(k \to 0, t) \to 1\ . \tag{12}$$

Here, t_i is some initial time when all scales are outside the Hubble radius, $k \ll aH$, and we can take $t_i = t_e$.

The problem in evaluating $\alpha(k)$ comes from the evolution of the perturbations for scales k' inside the Hubble radius, $k = (aH)|_{t_k} \le k' \le k_e$, or equivalently $1 \le x \le \frac{k_e}{k}$. This small scale evolution is encoded in the transfer function $T(k, t)$ which is defined through relation (12). For scales outside the horizon, the transfer function equals one. We emphasize that in (11), the transfer function must be taken *at the time* t_k of interest, not today. Clearly, an accurate value for (11) can be obtained only numerically and with an explicit knowledge of $T(k, t)$, which is of course model dependent [2].

Let us now compare our results with what is usually done using (6). The quantity $\sigma_H(t_0)$ is certainly finite, even for $n \ge 1$, because one must use in (2), according to (11), the actual power spectrum whose deformation on small scales is encoded in the present-day transfer function $T(k, t_0)$,

$$\sigma_H^2(t_0) = \delta_H^2(t_0) \int_0^{\infty} x^{n+2}\ W_{TH}^2(x)\ T^2(k_0 x, t_0)\ \mathrm{d}x\ , \tag{13}$$

where $k_0 \equiv (aH)_{t_0}$, and $\delta_H(t_0) \equiv \delta_H(k_0, t_0)$. The transfer function today, $T(k, t_0)$, effectively cuts the power on small scales and makes (13) finite. In addition, *today* $T(k, t_0)$ becomes already negligible for scales $k \ll k_e$; this is why the upper limit of integration in (13) can be replaced by infinity.

The ratio between $\sigma_H(t_0)$ and $\delta_H(t_0)$ often used (see e.g. [7, 6]),

$$\sigma_H(t_0) \approx 5\ \delta_H(t_0), \tag{14}$$

when simply substituted into (6), would yield the result

$$\sigma_H^2(t_k) \approx 25\ \delta_H^2(t_0)\ \left[\frac{M_H(t_0)}{M_H(t_{eq})}\right]^{\frac{n-1}{3}} \left[\frac{M_H(t_{eq})}{M_H(t_k)}\right]^{\frac{n-1}{2}} . \tag{15}$$

Eq. (15) is a reasonable first guess, but too inaccurate if one is willing to make precise predictions. According to (9) and (10), one must instead use for $t_k \ll t_{eq}$ the expression

$$\begin{aligned} \sigma_H^2(t_k) &= \frac{100}{81}\ \alpha^2(k)\ \delta_H^2(t_0)\ \left[\frac{M_H(t_0)}{M_H(t_{eq})}\right]^{\frac{n-1}{3}} \left[\frac{M_H(t_{eq})}{M_H(t_k)}\right]^{\frac{n-1}{2}} , \\ &= \frac{200}{9^3\pi^2}\ \alpha^2(k)\ k_0^3\ \Phi^2(k_0, t_0)\ \left[\frac{M_H(t_0)}{M_H(t_{eq})}\right]^{\frac{n-1}{3}} \left[\frac{M_H(t_{eq})}{M_H(t_k)}\right]^{\frac{n-1}{2}} . \end{aligned} \tag{16}$$

Eqs. (16), (10), and (11) are the main results of this section. Eq. (15) corresponds, in our notation, to $\alpha(k) = \alpha(k_0) = 5 \times 9/10 = 4.5$. It is not correct as $\alpha(k)$ is both model and scale dependent, since the transfer function depends on the cosmological parameters and the time t_k. Even for a given cosmological model for which (14) holds, it would certainly be incorrect at much earlier times t_k, according to (11). This can result in a large discrepancy in the computation of $\beta(M_H)$. It is recognized that the problem is much deeper than just a clever choice of the window function. Actually, the window function corresponding to the numerical results obtained in [16] is the top-hat window function. If one uses $\delta_{min} \approx 0.7$ as found by these authors, one must also use the top-hat window function in order to be consistent with their numerical results.

It is tacitly assumed in (16), (as in (6)), that the PBHs form before the time of equality, $t_k \ll t_{eq}$, and that the universe is first radiation-dominated and instantaneously becomes matter-dominated at t_{eq} (with $\Lambda = 0$). A more complicated evolution of the scale factor $a(t)$ will give a different expression. The observational input is the present density contrast $\delta_H(t_0)$ on the Hubble-radius scale which is found using the CMB anisotropy data.

What are the scales for which the discrepancy between (16) and (15) is most important? To get an idea, we consider the earliest formation times $t_k \sim t_{k_e}$, right after inflation. This gives

$$\alpha^2(k_e) \approx \int_0^1 x^{n+2}\ W_{TH}^2(x)\ \mathrm{d}x \approx 0.21\ , \tag{17}$$

with

$$0.202 \leq \alpha^2(k_e) \leq 0.218 \qquad \text{for} \qquad 1.3 \geq n \geq 1 \ . \tag{18}$$

Clearly there is a big difference for the earliest formation time $t_k \sim t_{k_e}$. A first rough estimate of $\alpha(k)$ corresponding to $M_H \approx 10^{15}$g gives $(\frac{10}{9})^2\ \alpha^2(M_H \approx 10^{15}\text{g}) \lesssim 2$, substantially smaller than 25. This estimate follows by taking into account that for modes outside the horizon one has $|\delta_k| \propto a^2$, while for modes within the horizon $|\delta_k|$ is roughly constant.

Interestingly, as seen from (5), the change implied by a smaller $\sigma_H(t_k)$ at earlier times, *amplifies* the change induced by a larger δ_{min} – one is led to a smaller $\beta(M)$. Therefore, constraints on n_{max} from PBH formation both on mass scales $M < 10^{15}$g (these concern PBHs that have already evaporated by today) as on scales $M > 10^{15}$g (see below) will be *less constraining* if one uses (16) instead of (15). We note finally that an effective quantum-to-classical transition takes place already for PBHs with masses $M \ll 10^{15}$g , but not for PBHs with $M \sim 1$g corresponding to $k \sim k_e$ [20] (with the exception of PBHs possibly produced during preheating, see e.g. [9, 10]). At these formation times, the growing and decaying modes of the fluctuations that reenter the Hubble radius are still of the same order (no squeezing). Therefore, the classical approach must be amended. Anyway, a correct expression of the power spectrum must then take the decaying mode into account [21].

2.3 Inflationary spectra with a scale

It is also our intention to investigate what happens when the assumption of a scale-free spectrum, tacitly or implicitly done in the literature, is dropped. We therefore consider spectra with a characteristic scale for which (9) and therefore (16) do not apply anymore. For each specific spectrum we have to compute $\sigma_H(t_k)$ numerically, but it is possible to give a rough estimate of what will happen if we consider some fiducial cases. Consider for example the case of a spectrum with a pure step in $\sigma_H(t_k)$ at $k = k_s$, with $t_{k_s} < t_{eq}$, where the ratio of $\sigma_H(t_k)$ on large scales to that on small scales is given by p. In this case, we have

$$\sigma_H(t_k) = \frac{10}{9}\ \alpha(k)\ \delta_H(t_0) \left[\frac{M_H(t_0)}{M_H(t_{eq})}\right]^{\frac{n-1}{6}} \left[\frac{M_H(t_{eq})}{M_H(t_k)}\right]^{\frac{n-1}{4}} \times \begin{cases} 1 & \text{for } k < k_s \\ p^{-1} & \text{for } k \geq k_s \end{cases} \ . \tag{19}$$

We emphasize that, as explained in Sect. 2, this does not correspond to a pure step in the primordial power spectrum $\delta_H(t_k)$. But with the approximation of a constant α for all PBH masses greater than 10^{15}g, this is certainly the simplest extension to (16) that has a characteristic scale. Here we shall restrict ourselves to this case. In a future paper we shall then consider models with a characteristic scale which are motived by an underlying Lagrangian, or even features in the inflaton potential, for example a particular primordial fluctuations spectrum which derives from a jump in the inflaton potential derivative. The resulting fluctuations spectrum for such a potential is of a universal form and an exact analytical expression has been derived by Starobinsky [13]. This expression depends (in addition to the overall normalization) on two parameters p and k_s, which have a similar meaning to those in the toy model considered here.

In our future work we will also take into account the effect of a cosmological constant corresponding to $\Omega_\Lambda \approx 0.7$ at present, since this seems to be strongly supported by recent observations and makes our current comprehension of the universe converge into a coherent picture.

3 Observational constraints

There are various limits on the initial mass fraction $\beta(M)$ of PBHs [7], most of them being related to the effects of Hawking radiation. Here, we restrict ourselves to the gravitational constraint

$$\Omega_{PBH,0} = 2\ \Omega_{PBH,eq}\ \Omega_{m,0} < 1\ , \tag{20}$$

which states that the present PBH mass density must not exceed the present density of the universe. In (20), $\Omega_{m,0}$ refers to *all* dustlike matter today including a possible contribution from PBHs. In order to relate (20) to the initial PBH abundance, one needs in principle to know the entire history of the universe right from the time when a mass scale M crosses into the horizon. Then one can compare the evolution of the PBH density ρ_{PBH} (which goes like $\rho_{PBH} \propto a^{-3}$) to the total density ρ_{tot}. In a radiation dominated universe $\rho_{tot} \propto a^{-4}$, so a small initial fractional PBH density can grow very large until today. We take again the evolution of the scale factor corresponding to (16). We follow steps similar to [7], but take into account all g-factors, i.e., use the constancy of the entropy $S = g_{*S}(T)a^3T^3$. This leads to the limit

$$\beta(M_H) < 1.57 \times 10^{-17} \left(\frac{M_H}{10^{15}\mathrm{g}}\right)^{\frac{1}{2}}\ h^2, \tag{21}$$

where we have used $t_{eq} = 4.36 \times 10^{10}\ \mathrm{s}\,(\Omega_{m,0}h^2)^{-2} \times (\frac{T_\gamma}{2.75})^6$ [22]. Strictly speaking, the behaviour of the scale factor is slightly more sophisticated, but we leave that aside here. Anyway, as we will show in Fig.1, a small change in (21) does not matter. Finally, since black holes of initial mass $M_{PBH} < 5 \times 10^{14}$ g will have evaporated by the present day, the tightest constraint is obtained for $M_H \sim 10^{15}$ g.

3.1 Scale-free spectrum

In order to compare the constraints on the spectral index with earlier results we first consider a scale-free spectrum, $n =$ const, on all scales. We further make the approximation $\alpha(k) \approx \alpha(k_0) \approx 4.5$ in order to compare our results with previous work. As we have seen in Sect. II, this is not correct. The exact numbers can

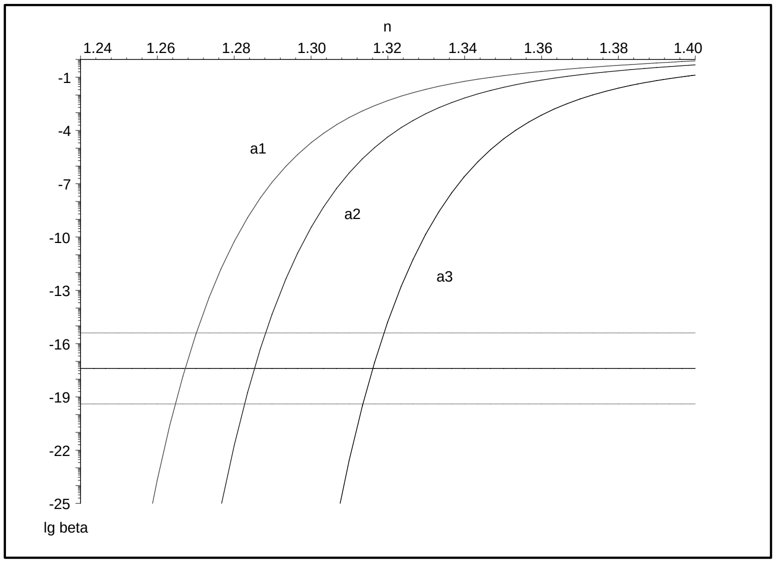

Figure 1: The quantity $\beta \equiv \beta(M = 10^{15}\text{ g})$ is shown as a function of the spectral index n, for several values of $\alpha \equiv \alpha(M = 10^{15}\text{ g})$. The straight lines represent the gravitational constraint (21) and, for the purpose of illustration, the same constraint weakened and strengthened by a factor of 100, respectively. The resulting maximally allowed value for n clearly depends only weakly on the precise value of the prefactor of the gravitational constraint (21), whereas it depends rather sensitively on the value of α. The value usually used in the literature, $\alpha \approx 4.5$, overestimates the initial PBH abundance significantly and thus leads to much stronger constraints on n (i.e. $n < 1.27$) than would be expected for the more realistic choice of $\alpha \approx 1.5$ (see section 2.2), which results in $n \lesssim 1.32$.

only be calculated numerically. This is beyond the scope of our paper, but we shall get an idea about the changes by displaying the dependence of the obtained constraints on the used value for $\alpha(k)$, see Fig. 1.

The data obtained from COBE can be used [7] to normalize the spectrum (6) to

$$\sigma_H(M_{H,0} \sim 10^{56}\text{ g}) = 9.5 \times 10^{-5}\ . \tag{22}$$

The above numbers are understood as to include the factor 10/9. In this way, the comparison with earlier results is straightforward. With the following setting of the parameters,

$$h := 0.5, \qquad \delta_{min} := \frac{1}{3}, \tag{23}$$

and by using equations (5) and (21), evaluated at $M_H = 10^{15}$ g, one gets for the spectrum (6) the following upper bound on the spectral index:

$$n < 1.27. \tag{24}$$

This result differs from $n < 1.31$ [7] and is much closer to the constraints they found due to the evaporation of PBHs, $n < 1.24$. This justifies our approach of only looking at the gravitational constraint - particularly since the details of black hole evaporation are still quite speculative. As mentioned above, the real constraints (i.e., taking into account the full expression (11) for α) are expected to be somewhat weaker, cf. Fig. 1.

We also had a look at the dependence of the result on the particular choice of the parameters (23). What we found was a weak dependence on h and M_H (i.e. the minimal mass to which the gravitational constraint (21) applies): varying h from 0.1 to 1 as well as varying M_H from 10^{14} g to 10^{16} g changes the constraint on n by only about 0.01.

The motivation for varying M_H is twofold: first, it amounts to the fact that the details of black hole evaporation are not yet fully understood, so the minimal mass of PBHs that have not evaporated by the present day is afflicted with some (small) uncertainty; second, if the mass of the PBH formed is not equal to the horizon mass M_H, but somewhat smaller, i. e. $M_{PBH} = \epsilon M_H, \epsilon < 1$, the minimal horizon mass to which the gravitational constraint (21) applies, is changed to $M_H = \epsilon^{-1} \cdot 10^{15}$ g. Semianalytical considerations, for example, require $\epsilon = \gamma^{\frac{3}{2}}$ [4]. In the radiation dominated case ($\gamma = \frac{1}{3}$) this would weaken the constraint (24) by only about 0.005. But even $\epsilon = 0.001$ weakens it by only slightly more than 0.02!

On the other hand, we have found, as expected, the dependence on the value of δ_{min} to be rather strong: for $\delta_{min} = 0.2$ one gets $n < 1.24$ and for $\delta_{min} = 0.7$ one gets $n < 1.30$. The latter point is rather important, since $\delta_{min} = \frac{1}{3}$ (as assumed above) relies on semianalytical arguments, whereas numerical analysis seems to support $\delta_{min} \approx 0.7$ [16]. Fig. 2 shows the dependence of n_{max} on δ_{min}.

3.2 Scale-free spectrum with a superimposed step

Next we consider the case of a simple extension to the scale-free spectrum, i.e. a power-law spectrum with a step as described by (19). Again, we use the approximation $\alpha(k) \approx \alpha(k_0) \approx 4.5$ in order to compare our results with the previous subsection. We restrict ourselves to the case $p \leq 1$, i.e. more power on smaller scales. The opposite case leads to interpretational problems for the quantity $\beta(M)$: For $p > 1$ it is no longer a monotonically falling function of mass and therefore cannot be interpreted anymore as the fractional mass density of PBHs of mass larger than M. For such a case, the Press-Schechter formalism would have to be modified. But $p < 1$ is the physically interesting case for us, since it would produce *more* instead of less PBHs on smaller scales.

In contrast to the scale-free case we now have three parameters - n, p and k_s (or $M_H(t_{k_s})$, respectively) - to be determined. One would therefore generally expect that the constraint (21) on $\beta(M)$ gives a plane in this parameter space. Since this constraint is a rising and $\beta(M)$ a falling function of mass, the strongest constraint will as before come from $M \sim 10^{15}$ g. But as long as $M_H(t_{k_s}) > 10^{15}$ g, $\beta(M \sim 10^{15}$ g) does not depend on the choice of k_s, so we get a functional

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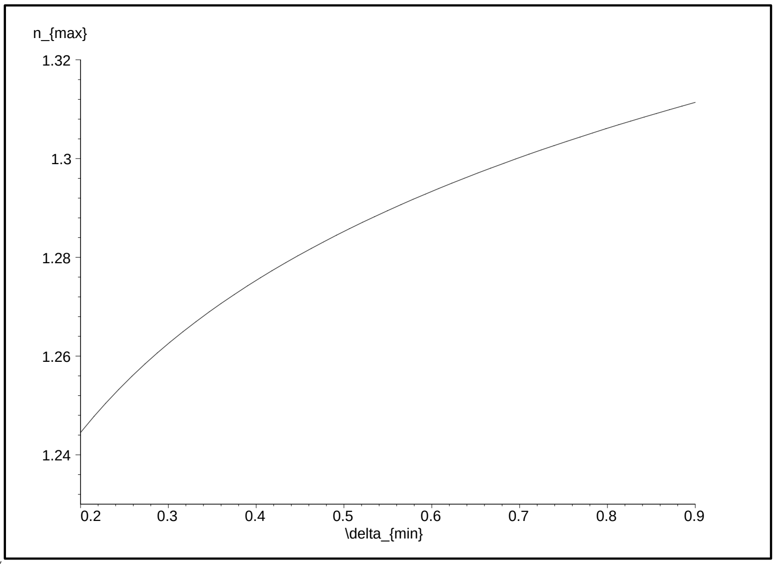

Figure 2: Dependence of the constraint on the spectral index, n_{max}, on δ_{min} for the case of a scale-free spectrum, with $h = 0.5, M_H = 10^{15}$ g. The constraint on n is clearly weakened for larger values of δ_{min}.

dependence between n_{max} and p only. In order to illustrate this point, Fig. 3 shows $\beta(M)$ together with the gravitational constraint (21).

Applying the same calculations as described in the previous paragraph, one gets $n_{max}(p)$ as depicted in Fig. 4. For $p = 1$ the result of the previous subsection is recovered (i.e. $n_{max} = 1.27$), but for lower values of p the constraint on n may be considerably strengthened. For example, $p = 0.01$ requires $n < 1.06$! Of course this result can also be used the other way around: if $n > 1.06$ were found by some experimental measure, the gravitational constraint on the PBH abundance would require $p > 0.01$.

The analysis sketched above does not tell us anything about the scale k_s at which the step occurs. It would nevertheless be interesting to learn something about k_s. This could arise, for example, from observations of MACHOs, hinting at the presence of a feature in $\beta(M)$ in the mass range $M > 10^{15}$g.

4 Conclusions

We have considered two aspects of the formation of PBHs. On the one hand, we have derived a more precise expression for the PBH abundance. It corrects the usual formula used, and could in principle substantially improve the accuracy of the obtained results. The amount of correction implied would need numerical calculation and will be presented elsewhere. Our results concerning this part of the present work are summarized in (16), (10), and (11). On the other hand, we have considered simple models where the spectrum is not scale-free and have compared it to the scale-free case. In the latter case we have found a somewhat stronger constraint than in [7] and have discussed the dependence of the result on the value of δ_{min}. In the former case, we have found that the presence of a step in the mass variance can drastically strengthen the constraints on n, giving in turn a tool to constrain the height of the step. This study can be extended to models with an underlying miscroscopic Lagrangian or phenomenologically motivated features in the potential. We plan to address these issues in a future publication.

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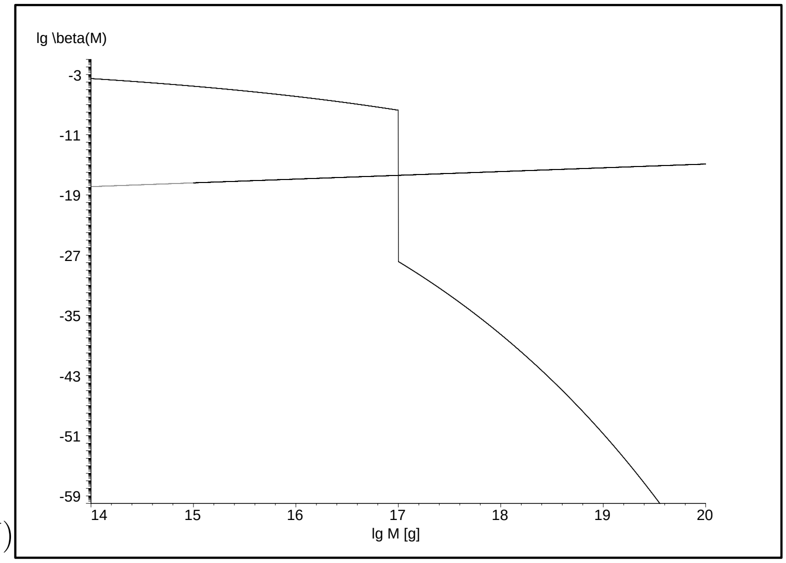
beta(M)beta(M)beta(M)lg $\beta(M)$ 

Figure 3: This figure shows $\beta(M)$ for a step spectrum as in (19), with (arbitrarily chosen) $n = 1.27$, $M_H(t_{k_s}) = 10^{17}$ g and $p = 0.5$. The straight line is the gravitational constraint (21), which applies only for $M \gtrsim 10^{15}$ g. The figure illustrates that the gravitational constraint is to be evaluated at 10^{15} g and that the result does not depend on the choice of k_s (as long as $M_H(t_{k_s}) > 10^{15}$ g).

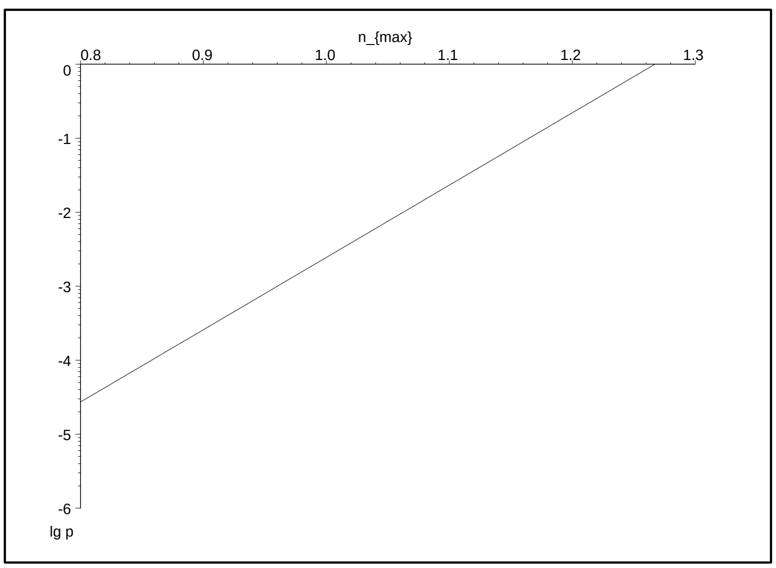


Figure 4: Dependence of n_{max} on p for the step spectrum (19). For $p = 1$ the result for the scale-free case is recovered. For $p < 1$ the constraint on n is considerably strengthened.